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REGULAR PERTURBATION METHODS FOR A REGION WITH A CRACK

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The paper considers a model problem for Poisson's equation for a region containing a crack or a set of cracks under arbitrary linear perturbation. Variational formulation of the problem using smooth mapping of regions yields a complete asymptotic expansion of the solution in the perturbation parameter, which is a generalized shape derivative. This global asymptotic expansion of the solution was used to derive representations of arbitrary-order derivatives for the potential energy function, stress intensity factors, and invariant energy integrals in general form and for basis perturbations of the region (shear, tension, and rotation). The problem of the local growth of a branching crack for the Griffith fracture criterion and the linearized problem of optimal location of a rectilinear crack in a body with the energy function as a cost functional were formulated.

INTRODUCTION

Singular perturbation theory is commonly used to study regions with nonsmooth boundaries, in particular, regions with cracks (see, for example, [1–3]). Procedures of justifying asymptotic representations are given in [4]. Mapping of a region with a crack based on smooth coordinate transformation reduces the problem to regular perturbations even for regions with nonsmooth boundaries. In this case, the crack problem is considered in a variational formulation, which allows this approach to be applied to the general problem, including the nonlinear crack problem under nonpenetration conditions on the sides [5–7]. This approach generalizes the methods for optimizing the shape of smooth regions [8, 9] located inside regions with nonsmooth boundaries due to cracks. The methods proposed can be used to obtain invariant energy integrals (Cherepanov–Rice integral, etc. [10–13]) as a particular case of the general perturbation of a cracked region. Numerical solution of the crack growth problem using the formulas obtained is described in [14].

1. LINEAR PERTURBATION OF A REGION WITH A CRACK

1.1. Formulation of the Problem. Let us consider a region $\Omega \subset \mathbb{R}^2$ with a Lipschitzian continuous boundary $\partial\Omega$ and a part of this region $\Gamma_{\mathcal{D}} \subseteq \partial\Omega$ with $\operatorname{meas}\Gamma_{\mathcal{D}} > 0$. Let Ω contain a crack defined by a finite set of Lipschitzian continuous curves Γ_0. Specifying a tangent vector $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and a normal vector $\boldsymbol{\nu} = (\nu_1, \nu_2)$ to Γ_0, we assume that the direction $\boldsymbol{\nu}$ corresponds to the crack side Γ_0^+ and the direction $-\boldsymbol{\nu}$ to the side Γ_0^-. We define the region with the crack as $\Omega_0 = \Omega \setminus \overline{\Gamma}_0$ with the boundary $\partial\Omega_0 = \partial\Omega \cup \overline{\Gamma}_0^+ \cup \overline{\Gamma}_0^-$. We assume that Ω_0 is an open connected set in $\mathbb{R}^2$, and the curves comprising Γ_0 can be continued until they intersect the part of the outer boundary $\Gamma_{\mathcal{D}}$ at a finite angle, without intersecting with one another. The first condition eliminates self-intersection of the curves and allows standard Sobolev spaces to be defined in Ω_0; the second condition provides for satisfaction

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of the Korn inequality in Ω_0. This construction simulates the following crack geometries in a body: one crack (probably containing corners or reaching the boundary); a family of disjoint cracks; a branching crack (i.e., a set of cracks intersecting only at one point at a nonzero angle); a combination of the geometries listed.

Let the external loading be specified by a smooth function $f \in C^\infty([-\varepsilon_0, \varepsilon_0] \times \mathbb{R}^2)$. We introduce a Sobolev space $\tilde{H}^1(\Omega_0) = \{v \in H^1(\Omega_0),\ v = 0 \text{ a.e. } \Gamma_{\mathcal{D}}\}$ with the homogeneous Dirichlet condition on the part of the external boundary $\Gamma_{\mathcal{D}}$. We consider the following model problem in the variational formulation for Poisson's equation in the region with a crack:

$$\int\limits_{\Omega_0} \nabla u^0 \cdot \nabla v = \int\limits_{\Omega_0} f(0) v \qquad \forall\ \ v \in \tilde{H}^1(\Omega_0). \tag{1}$$

According to the general theory of solvability of variational problems and by virtue of the Korn inequality, there exists a unique solution $u^0 \in \tilde{H}^1(\Omega_0)$ of problem (1) characterized by the following relations that are valid in the region Ω_0 and on the crack:

$$-\Delta u^0 = f(0) \qquad \text{in} \quad \Omega_0; \tag{2}$$

$$\frac{\partial u^0}{\partial \boldsymbol{\nu}} = 0 \qquad \text{on} \quad \Gamma_0^{\pm}. \tag{3}$$

Because f is assumed to be smooth, from (2) it follows that the solution u^0 will be smooth inside the region Ω_0 up to the regular part of the boundary $\partial\Omega_0$. The potential energy in problem (1) can be written as

$$\mathcal{P}(0) = \frac{1}{2} \int\limits_{\Omega_0} |\nabla u^0|^2 - \int\limits_{\Omega_0} f(0) u^0$$

or, using equality (1) with $v = u^0$, in equivalent form

$$\mathcal{P}(0) = -\frac{1}{2} \int\limits_{\Omega_0} f(0) u^0. \tag{4}$$

For a fixed small parameter $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we consider a perturbation $\boldsymbol{\Phi}^\varepsilon$ of the region Ω_0 with the crack Γ_0 that defines a perturbed region $\Omega_\varepsilon = \boldsymbol{\Phi}^\varepsilon(\Omega_0)$ with a perturbed crack $\Gamma_\varepsilon = \boldsymbol{\Phi}^\varepsilon(\Gamma_0)$. It is assumed that Ω_ε and Γ_ε have the same properties as Ω_0 and Γ_0 and meas $\boldsymbol{\Phi}^\varepsilon(\Gamma_{\mathcal{D}}) > 0$. We use a linear perturbation in the form $\boldsymbol{\Phi}^\varepsilon(\boldsymbol{x}) = \boldsymbol{x} + \varepsilon \boldsymbol{\Phi}(\boldsymbol{x})$ with the specified function $\boldsymbol{\Phi} = (\Phi_1, \Phi_2)$, where $\Phi_i \in W^{1,\infty}(\mathbb{R}^2)$ $(i = 1, 2)$, or, in coordinate form,

$$y_1 = x_1 + \varepsilon \Phi_1(\boldsymbol{x}), \quad y_2 = x_2 + \varepsilon \Phi_2(\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, x_2) \in \Omega_0, \quad \boldsymbol{y} = (y_1, y_2) \in \Omega_\varepsilon. \tag{5}$$

Let us formulate the variational problem for Poisson's equation in the perturbed region Ω_ε:

$$\int\limits_{\Omega_\varepsilon} \nabla u^\varepsilon \cdot \nabla v = \int\limits_{\Omega_\varepsilon} f(\varepsilon) v \qquad \forall\ v \in \tilde{H}^1(\Omega_\varepsilon). \tag{6}$$

Here $\tilde{H}^1(\Omega_\varepsilon) = \{v \in H^1(\Omega_\varepsilon),\ v = 0 \text{ a.e. } \boldsymbol{\Phi}^\varepsilon(\Gamma_{\mathcal{D}})\}$. Similarly to problem (1), problem (6) has a unique solution $u^\varepsilon \in \tilde{H}^1(\Omega_\varepsilon)$. Thus, we obtain a one-parameter family of problems (6), which depend on the perturbation parameter ε, and Eq. (1) is a particular case of (6) for $\varepsilon = 0$. The corresponding function of the potential energy $\mathcal{P}:\ (-\varepsilon_0, \varepsilon_0) \mapsto \mathbb{R}$ is in similar form to (4):

$$\mathcal{P}(\varepsilon) = -\frac{1}{2} \int\limits_{\Omega_\varepsilon} f(\varepsilon) u^\varepsilon. \tag{7}$$

1.2. Global Asymptotic Expansion of the Solution. We calculate the functional matrix of transformation (5)

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \begin{pmatrix} 1 + \varepsilon \Phi_{1,1} & \varepsilon \Phi_{1,2} \\ \varepsilon \Phi_{2,1} & 1 + \varepsilon \Phi_{2,2} \end{pmatrix},$$

whose determinant is equal to

$$J(\varepsilon) = 1 + \varepsilon \operatorname{div} \boldsymbol{\Phi} + \varepsilon^2 \Big| \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{x}} \Big|. \tag{8}$$

Here $\operatorname{div}\boldsymbol{\Phi} = \Phi_{1,1} + \Phi_{2,2}$ and $|\partial\boldsymbol{\Phi}/\partial\boldsymbol{x}| = \Phi_{1,1}\Phi_{2,2} - \Phi_{1,2}\Phi_{2,1}$. For sufficiently small ε, $J(\varepsilon) > 0$ almost everywhere in Ω_0, the mapping of $\boldsymbol{\Phi}^\varepsilon$ is biunivocal, and its inverse mapping $(\boldsymbol{\Phi}^\varepsilon)^{-1}$ exists. In this case, $(\boldsymbol{\Phi}^\varepsilon)^{-1}(\Omega_\varepsilon) = \Omega_0$ and $(\boldsymbol{\Phi}^\varepsilon)^{-1}(\Gamma_\varepsilon) = \Gamma_0$. For an arbitrary function $v \in \tilde{H}^1(\Omega_\varepsilon)$, we have $v \circ \boldsymbol{\Phi}^\varepsilon \in \tilde{H}^1(\Omega_0)$, where

$$(v \circ \boldsymbol{\Phi}^\varepsilon)(\boldsymbol{x}) \equiv v(\boldsymbol{x} + \varepsilon\boldsymbol{\Phi}(\boldsymbol{x})) \qquad (\boldsymbol{x} \in \Omega_0) \tag{9}$$

by virtue of the biunivocality of the mapping $\boldsymbol{\Phi}^\varepsilon$ and the boundedness of the first-order derivatives of $\boldsymbol{\Phi}$. The inverse statement is also valid: from $v \in \tilde{H}^1(\Omega_0)$, it follows that $v \circ (\boldsymbol{\Phi}^\varepsilon)^{-1} \in \tilde{H}^1(\Omega_\varepsilon)$. The inverse functional matrix is written as

$$\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{y}} = \begin{pmatrix} 1 - \varepsilon\Phi_{1,1}/J(\varepsilon) - \varepsilon^2|\partial\boldsymbol{\Phi}/\partial\boldsymbol{x}|/J(\varepsilon) & -\varepsilon\Phi_{1,2}/J(\varepsilon) \\ -\varepsilon\Phi_{2,1}/J(\varepsilon) & 1 - \varepsilon\Phi_{2,2}/J(\varepsilon) - \varepsilon^2|\partial\boldsymbol{\Phi}/\partial\boldsymbol{x}|/J(\varepsilon) \end{pmatrix},$$

from which we have the gradient transformation

$$\nabla \circ \boldsymbol{\Phi}^\varepsilon = \nabla : \frac{\partial\boldsymbol{x}}{\partial\boldsymbol{y}} = \nabla - \frac{\varepsilon}{J(\varepsilon)}\nabla : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}} - \frac{\varepsilon^2}{J(\varepsilon)}\left|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\right|\nabla. \tag{10}$$

Here $\nabla : \partial\boldsymbol{\Phi}/\partial\boldsymbol{x} = (\boldsymbol{\Phi}_{,1} \cdot \nabla, \boldsymbol{\Phi}_{,2} \cdot \nabla)$. Applying the coordinate transformation $\boldsymbol{\Phi}^\varepsilon$ to the integrals in problem (6) and using Eq. (10), we obtain

$$\int\limits_{\Omega_0} J(\varepsilon)\Big(\nabla(u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon) - \frac{\varepsilon}{J(\varepsilon)}\nabla(u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon) : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}} - \frac{\varepsilon^2}{J(\varepsilon)}\left|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\right|\nabla(u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon)\Big) \cdot \Big(\nabla v - \frac{\varepsilon}{J(\varepsilon)}\nabla v : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}} - \frac{\varepsilon^2}{J(\varepsilon)}\left|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\right|\nabla v\Big)$$

$$= \int\limits_{\Omega_0} J(\varepsilon)(f(\varepsilon) \circ \boldsymbol{\Phi}^\varepsilon)v \quad \forall\, v \in \tilde{H}^1(\Omega_0). \tag{11}$$

Using the Hölder and Korn inequalities in Eq. (11) with $v = u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon$, we can prove the following uniform estimate for sufficiently small ε:

$$\|u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon\|_{H^1(\Omega_0)} \leqslant \text{const.} \tag{12}$$

Thus, the following theorem is valid.

Theorem 1. *For sufficiently small ε, the solution of problem* (6) *mapped by $\boldsymbol{\Phi}^\varepsilon$ from* (5) *onto a fixed region Ω_0 is a unique solution of problem* (11).

We use representation (8) to obtain the series expansion

$$\frac{1}{J(\varepsilon)} = \sum_{n=0}^{\infty} \varepsilon^n J_n(\boldsymbol{\Phi}), \qquad J_n(\boldsymbol{\Phi}) = \sum_{k=0}^{[n/2]} \frac{(-1)^{n-k}(n-k)!}{k!(n-2k)!}(\operatorname{div}\boldsymbol{\Phi})^{n-2k}\left|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\right|^k. \tag{13}$$

Then, in accordance with (8) and (13), the operator on the left side of (11) admits the asymptotic expansion

$$J(\varepsilon)\Big(\nabla u - \frac{\varepsilon}{J(\varepsilon)}\nabla u : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}} - \frac{\varepsilon^2}{J(\varepsilon)}\left|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\right|\nabla u\Big) \cdot \Big(\nabla v - \frac{\varepsilon}{J(\varepsilon)}\nabla v : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}} - \frac{\varepsilon^2}{J(\varepsilon)}\left|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\right|\nabla v\Big)$$

$$= \nabla u \cdot \nabla v + \sum_{n=1}^{\infty} \varepsilon^n A_n(\boldsymbol{\Phi}; u, v), \tag{14}$$

where the bilinear and symmetric (about u and v) forms of A_n are written as

$$A_1(\boldsymbol{\Phi}; u, v) = \operatorname{div}\boldsymbol{\Phi}\nabla u \cdot \nabla v - \Big(\nabla u : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big) \cdot \nabla v - \nabla u \cdot \Big(\nabla v : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big), \tag{15}$$

$$A_2(\boldsymbol{\Phi}; u, v) = -\left|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\right|\nabla u \cdot \nabla v + \Big(\nabla u : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big) \cdot \Big(\nabla v : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big) \tag{16}$$

or, for $n = 3, 4, \ldots$,

$$A_n(\boldsymbol{\Phi}; u, v) = J_{n-2}(\boldsymbol{\Phi})\Big(\nabla u : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big) \cdot \Big(\nabla v : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big)$$

$$+ J_{n-3}(\boldsymbol{\Phi})\left|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\right|\Big[\Big(\nabla u : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big) \cdot \nabla v + \nabla u \cdot \Big(\nabla v : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big)\Big] + J_{n-4}(\boldsymbol{\Phi})\left|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\right|^2 \nabla u \cdot \nabla v$$

if we formally set $J_{-1}(\boldsymbol{\Phi}) = 0$. The forms of A_n are combinations of the first-order derivatives of u and v and the coefficients including the powers of the first-order derivatives of $\boldsymbol{\Phi}$; therefore, they are defined correctly for $u, v \in \tilde{H}^1(\Omega_0)$ and $\boldsymbol{\Phi} \in [W^{1,\infty}(\mathbb{R}^2)]^2$.

Representation (9) and the smoothness of f lead to the obvious series expansion

$$f(\varepsilon)\circ\mathbf{\Phi}^\varepsilon=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}f_n(\mathbf{\Phi}),\qquad f_n(\mathbf{\Phi})=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\,\frac{\partial^{n-k}}{\partial\mathbf{\Phi}^{n-k}}\Big(\frac{\partial^k f}{\partial\varepsilon^k}(0)\Big),\tag{17}$$

where $\partial/\partial\mathbf{\Phi}=\mathbf{\Phi}\cdot\nabla$ and $\partial^m/\partial\mathbf{\Phi}^m=\sum\limits_{k=0}^{m}\Phi_1^k\Phi_2^{m-k}\partial^m/\partial x_1^k\partial x_2^{m-k}$ $(m=2,3,\ldots)$. Multiplying (17) by (8), we obtain the asymptotic expansion of the function on the right side of Eq. (11)

$$J(\varepsilon)(f(\varepsilon)\circ\mathbf{\Phi}^\varepsilon)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\,F_n(\mathbf{\Phi}),\qquad F_0(\mathbf{\Phi})=f(0)\tag{18}$$

with the expansion coefficients

$$F_1(\mathbf{\Phi})=\frac{\partial f}{\partial\varepsilon}(0)+\operatorname{div}(\mathbf{\Phi}f(0)),\tag{19}$$

$$F_2(\mathbf{\Phi})=\frac{\partial^2 f}{\partial\varepsilon^2}(0)+2\operatorname{div}\Big(\mathbf{\Phi}\,\frac{\partial f}{\partial\varepsilon}(0)\Big)+2\Big|\frac{\partial\mathbf{\Phi}}{\partial\boldsymbol{x}}\Big|\,f(0)+2\operatorname{div}\mathbf{\Phi}\,\frac{\partial f(0)}{\partial\mathbf{\Phi}}+\frac{\partial^2 f(0)}{\partial\mathbf{\Phi}^2}\tag{20}$$

or, for $n=2,3,\ldots$,

$$F_n(\mathbf{\Phi})=f_n(\mathbf{\Phi})+n\,\operatorname{div}\mathbf{\Phi}\,f_{n-1}(\mathbf{\Phi})+n(n-1)\Big|\frac{\partial\mathbf{\Phi}}{\partial\boldsymbol{x}}\Big|\,f_{n-2}(\mathbf{\Phi}).$$

Following [9], let us construct an asymptotic expansion of the solution of problem (11) in ε

$$u^\varepsilon\circ\mathbf{\Phi}^\varepsilon=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\overset{(n)}{u}(\mathbf{\Phi}),\qquad \overset{(0)}{u}(\mathbf{\Phi})=u^0\quad\text{in}\quad \tilde H^1(\Omega_0)\tag{21}$$

with the corresponding derivatives $\overset{(1)}{u}(\mathbf{\Phi})\equiv\dot u(\mathbf{\Phi})$, $\overset{(2)}{u}(\mathbf{\Phi})\equiv\ddot u(\mathbf{\Phi})$, etc. We call these derivatives global derivatives of the solution of the corresponding order because expansion (21) is sought over the entire initial region Ω_0, unlike the local expansion inside Ω_0, which is defined below. We also treat them as complete derivatives of the perturbed solution $u^\varepsilon(\boldsymbol{x}+\varepsilon\mathbf{\Phi})$ over ε in the general sense. Using the terminology of shape optimization in smooth regions [8], we conclude that for f that is independent of ε, the first global derivative $\dot u(\mathbf{\Phi})$ is a strong material derivative of the solution in the direction of the velocity field $\mathbf{\Phi}$.

We substitute formally expansion (21) into Eq. (11) and use expansions (14) and (18). As a result, collecting multipliers at identical powers of ε, we establish that the global derivatives are determined from the problems

$$\int\limits_{\Omega_0}\nabla\dot u(\mathbf{\Phi})\cdot\nabla v=\int\limits_{\Omega_0}[F_1(\mathbf{\Phi})v-A_1(\mathbf{\Phi};u^0,v)]\qquad\forall\ v\in\tilde H^1(\Omega_0);\tag{22}$$

$$\int\limits_{\Omega_0}\nabla\ddot u(\mathbf{\Phi})\cdot\nabla v=\int\limits_{\Omega_0}[F_2(\mathbf{\Phi})v-2A_1(\mathbf{\Phi};\dot u(\mathbf{\Phi}),v)-2A_2(\mathbf{\Phi};u^0,v)]\tag{23}$$

or, for $n=3,4,\ldots$, from the problem

$$\int\limits_{\Omega_0}\nabla\overset{(n)}{u}(\mathbf{\Phi})\cdot\nabla v=\int\limits_{\Omega_0}\Big(F_n(\mathbf{\Phi})v-\sum_{k=1}^{n}\frac{n!}{(n-k)!}\,A_k(\mathbf{\Phi};\overset{(n-k)}{u}(\mathbf{\Phi}),v)\Big).\tag{24}$$

Problems (22)–(24), together with problem (1), formulate an iterative scheme for successive determination of the functions $\overset{(0)}{u}(\mathbf{\Phi})=u^0$, $\overset{(1)}{u}(\mathbf{\Phi})$, etc. [Formulas (22) and (23) follow from (24) for $n=1$ and 2.] Since the right side of (24) is a linear continuous functional over $\tilde H^1(\Omega_0)$, by virtue of the Korn inequality there exists a unique solution $\overset{(n)}{u}(\mathbf{\Phi})\in\tilde H^1(\Omega_0)$ of problem (24) for $n=1,2,\ldots$. Being similar to the initial problem (1), Eq. (24) differs from it only in the fictitious data inside the region and on the boundaries, which are found by the iterative method. Thus, we defined a procedure of generalized differentiation of the solution with respect to the region perturbation.

To prove the validity of expansion (21), we consider the corresponding estimates. Subtraction of Eq. (1) from Eq. (11) using (14) and (18) yields

$$\int\limits_{\Omega_0} \nabla(u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon - u^0) \cdot \nabla v = \varepsilon \int\limits_{\Omega_0} [\bar{F}_1(\boldsymbol{\Phi}) v - \bar{A}_1(\boldsymbol{\Phi}; u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon, v)],$$

where the bar denotes residual members in the corresponding expansions (14) and (18). If we set $v = u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon - u^0$ and use the Hölder and Korn inequalities, it is evident from estimate (12) that $\|u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon - u^0\|_{H^1(\Omega_0)} \leqslant c\,\varepsilon$. We then use induction. Let the following inequality be satisfied:

$$\Big\|u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon - \sum_{k=0}^{m} \frac{\varepsilon^k}{k!} \overset{(k)}{u}(\boldsymbol{\Phi})\Big\|_{H^1(\Omega_0)} \leqslant c\,\varepsilon^{m+1}, \qquad m = 0, \dots, n-1. \tag{25}$$

Constructing a partial sum of the nth order from Eqs. (1), (11), and (24) and using expansions (14) and (18), we obtain

$$\int\limits_{\Omega_0} \nabla\Big(u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon - \sum_{k=0}^{n} \frac{\varepsilon^k}{k!} \overset{(k)}{u}(\boldsymbol{\Phi})\Big) \cdot \nabla v = -\sum_{m=1}^{n} \varepsilon^m \int\limits_{\Omega_0} A_m\Big(\boldsymbol{\Phi}; u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon - \sum_{k=0}^{n-m} \frac{\varepsilon^k}{k!} \overset{(k)}{u}(\boldsymbol{\Phi}), v\Big)$$

$$+ \varepsilon^{n+1} \int\limits_{\Omega_0} \Big[\frac{1}{(n+1)!} \bar{F}_{n+1}(\boldsymbol{\Phi}) v - \bar{A}_{n+1}(\boldsymbol{\Phi}; u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon, v)\Big].$$

Again, substituting $v = u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon - \sum\limits_{k=0}^{n} \varepsilon^k \overset{(k)}{u}(\boldsymbol{\Phi})/k!$ and using the Hölder and Korn inequalities, we have the following equation from estimates (12) and (25):

$$\Big\|u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon - \sum_{k=0}^{n} \frac{\varepsilon^k}{k!} \overset{(k)}{u}(\boldsymbol{\Phi})\Big\|_{H^1(\Omega_0)} \leqslant c\,\varepsilon^{n+1}. \tag{26}$$

Thus, we proved the following theorem:

Theorem 2. *For a sufficiently small ε, there exist global derivatives of the solution $\overset{(n)}{u}(\boldsymbol{\Phi}) \in \tilde{H}^1(\Omega_0)$ of an arbitrary order n with respect to linear perturbation of the region, which are unique solutions of problem* (1) *for $n = 0$ and of problem* (24) *for $n = 1, 2, \dots$. In this case, the global asymptotic expansion* (21) *of the solution of problem* (11) *with estimate* (26) *is valid for an arbitrary n.*

1.3. Local Asymptotic Expansion of the Solution. The inverse coordinate transformation $(\boldsymbol{\Phi}^\varepsilon)^{-1}$ can be applied to equality (21), from which we obtain the representation

$$u^\varepsilon = u^0 \circ (\boldsymbol{\Phi}^\varepsilon)^{-1} + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \overset{(n)}{u}(\boldsymbol{\Phi}) \circ (\boldsymbol{\Phi}^\varepsilon)^{-1} \qquad \text{in} \quad \tilde{H}^1(\Omega_\varepsilon). \tag{27}$$

We consider a set $\overline{K}$ that is relatively compact inside Ω_0. For sufficiently small ε, the condition $\overline{K} \subset \Omega_\varepsilon$ is satisfied, and, hence, Eq. (27) is also valid in $H^1_{\text{loc}}(\Omega_0)$. At the same time, from the equilibrium equation (2) it follows that the solution u^0 of problem (1) is smooth inside Ω_0; therefore, $u^0 \circ (\boldsymbol{\Phi}^\varepsilon)^{-1}$ inside $\overline{K}$ can be expanded in a series in ε as follows. We assume that the perturbation function $\boldsymbol{\Phi}$ is smooth, i.e., $\boldsymbol{\Phi} \in [C^\infty(\mathbb{R}^2)]^2$. Differentiation of (5) as an implicit function yields the inverse function $\boldsymbol{x}(\varepsilon, \boldsymbol{y})$ as a solution of the nonlinear ordinary first-order differential equation for ε:

$$\frac{d\boldsymbol{x}}{d\varepsilon} = -\boldsymbol{\Phi}(\boldsymbol{x}) : \Big(\frac{\partial(\boldsymbol{x} + \varepsilon\boldsymbol{\Phi}(\boldsymbol{x}))}{\partial \boldsymbol{x}}\Big)^{-1}, \qquad \boldsymbol{x}\Big|_{\varepsilon=0} = \boldsymbol{y}. \tag{28}$$

Since $\boldsymbol{\Phi}$ is assumed to be smooth, a smooth solution of problem (28) exists that can be written as

$$\boldsymbol{x} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \boldsymbol{X}^n(\boldsymbol{y}), \qquad \boldsymbol{X}^0(\boldsymbol{y}) = \boldsymbol{y}, \quad \boldsymbol{X}^1(\boldsymbol{y}) = -\boldsymbol{\Phi}(\boldsymbol{y}). \tag{29}$$

Here the functions $\boldsymbol{X}^2, \boldsymbol{X}^3, \dots$ can be found by sequentially differentiating Eq. (28) with respect to ε, assuming that $\varepsilon = 0$ and $\boldsymbol{x} = \boldsymbol{y}$. For example, $\boldsymbol{X}^2(\boldsymbol{y}) = \Big[\boldsymbol{\Phi}\, \mathrm{div}\, \boldsymbol{\Phi} + \boldsymbol{\Phi} : (\partial\boldsymbol{\Phi}/\partial\boldsymbol{x} - |\partial\boldsymbol{\Phi}/\partial\boldsymbol{x}|(\partial\boldsymbol{\Phi}/\partial\boldsymbol{x})^{-1})\Big]_{\boldsymbol{x}=\boldsymbol{y}}$, etc. Series expansion of $u^0 \circ (\boldsymbol{\Phi}^\varepsilon)^{-1}$ in ε as a smooth function in $\overline{K}$ using (29) yields

$$u^0 \circ (\boldsymbol{\Phi}^\varepsilon)^{-1} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \overset{(0n)}{u}(\boldsymbol{\Phi}) \qquad \text{in} \quad H^1_{\text{loc}}(\Omega_0), \tag{30}$$

where $\overset{(00)}{u}(\boldsymbol{\Phi}) = u^0$, $\overset{(01)}{u}(\boldsymbol{\Phi}) = -\partial u^0/\partial \boldsymbol{\Phi}$, $\overset{(02)}{u}(\boldsymbol{\Phi}) = \partial^2 u^0/\partial \boldsymbol{\Phi}^2 + \boldsymbol{X}^2 \cdot \nabla u^0$, etc. This procedure can be applied to the following functions in representation (27). For $v \in C_0^\infty(\Omega_0)$, integration of (22) by parts yields the equation

$$-\Delta \dot{u}(\boldsymbol{\Phi}) = \frac{\partial f}{\partial \varepsilon}(0) + \boldsymbol{\Phi} \cdot \nabla f(0) - 2 \sum_{i=1}^{2} \nabla u^0_{,i} \cdot \boldsymbol{\Phi}_{,i} + \nabla u^0 \cdot \Delta \boldsymbol{\Phi} \qquad \text{in} \quad \Omega_0.$$

Therefore, by virtue of the assumption of smoothness of f and $\boldsymbol{\Phi}$ and local smoothness of u^0, the function $\dot{u}(\boldsymbol{\Phi})$ will also be smooth inside Ω_0. Similar reasoning for solutions $\overset{(n)}{u}(\boldsymbol{\Phi})$ of problem (24) for $n = 2, 3, \ldots$ leads us to the conclusion on the local smoothness of all $\overset{(n)}{u}(\boldsymbol{\Phi})$ inside the region Ω_0. Therefore, similarly to (30), we obtain the expansions

$$\overset{(n)}{u}(\boldsymbol{\Phi}) \circ (\boldsymbol{\Phi}^\varepsilon)^{-1} = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \overset{(nk)}{u}(\boldsymbol{\Phi}), \qquad n = 0, 1, \ldots \quad \text{in} \quad H^1_{\text{loc}}(\Omega_0). \tag{31}$$

Here $\overset{(n0)}{u}(\boldsymbol{\Phi}) = \overset{(n)}{u}(\boldsymbol{\Phi})$, $\overset{(n1)}{u}(\boldsymbol{\Phi}) = -\partial \overset{(n)}{u}(\boldsymbol{\Phi})/\partial \boldsymbol{\Phi}$, $\overset{(n2)}{u}(\boldsymbol{\Phi}) = \partial^2 \overset{(n)}{u}(\boldsymbol{\Phi})/\partial \boldsymbol{\Phi}^2 + \boldsymbol{X}^2 \cdot \nabla \overset{(n)}{u}(\boldsymbol{\Phi})$, etc. Substituting (31) into (27) and collecting terms at the same powers of ε, we obtain a local asymptotic expansion in ε [compare with (21)] of the perturbed solution

$$u^\varepsilon = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} u^{(n)}(\boldsymbol{\Phi}), \quad u^{(0)}(\boldsymbol{\Phi}) = u^0 \qquad \text{in} \quad H^1_{\text{loc}}(\Omega_0) \tag{32}$$

with local derivatives of the solution $u^{(n)}(\boldsymbol{\Phi})$ of the nth order, which are determined as distributions inside Ω_0 in the form

$$\begin{gathered} u^{(1)}(\boldsymbol{\Phi}) \equiv u'(\boldsymbol{\Phi}) = \dot{u}(\boldsymbol{\Phi}) - \boldsymbol{\Phi} \cdot \nabla u^0, \\ u^{(2)}(\boldsymbol{\Phi}) \equiv u''(\boldsymbol{\Phi}) = \ddot{u}(\boldsymbol{\Phi}) - 2\, \frac{\partial \dot{u}(\boldsymbol{\Phi})}{\partial \boldsymbol{\Phi}} + \boldsymbol{X}^2 \cdot \nabla u^0 + \frac{\partial^2 u^0}{\partial \boldsymbol{\Phi}^2} \end{gathered} \tag{33}$$

or, generally,

$$u^{(n)}(\boldsymbol{\Phi}) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, u^{(k(n-k))}(\boldsymbol{\Phi})$$

for $n = 0, 1, \ldots$.

In this case, the first-order local derivative in (33) is defined in $L^2(\Omega_0)$. These derivatives can be treated as partial derivatives of the perturbed solution u^ε with respect to the parameter ε. According to shape optimization theory for smooth regions [8], if f does not depend on ε and the first local derivative in (33) belongs to the class $\tilde{H}^1(\Omega_0)$, this is a strong shape derivative in the direction of the velocity field $\boldsymbol{\Phi}$. Generally, this is incorrect for a crack; therefore, it is impossible to determine the shape derivative as a solution of any variational problem. Thus, we proved the following theorem:

Theorem 3. *For a small ε and a smooth perturbation function $\boldsymbol{\Phi}$, the local asymptotic expansion* (32) *of the perturbed solution u^ε of problem* (6) *with local derivatives of the solution $u'(\boldsymbol{\Phi})$, $u''(\boldsymbol{\Phi})$, $\ldots$ with respect to the linear perturbation of the region is valid.*

Because the solution u^ε of problem (6) is defined in the perturbed region Ω_ε and, hence, does not depend on the choice of the perturbed function $\boldsymbol{\Phi}$, the derivatives in expansion (32) in this sense do not depend on $\boldsymbol{\Phi}$.

1.4. Asymptotic Expansion of the Potential Energy. The integral representation of the potential energy (7) that follows from the above definition is subjected to the coordinate transformation $\boldsymbol{\Phi}^\varepsilon$:

$$\mathcal{P}(\varepsilon) = -\frac{1}{2} \int_{\Omega_0} J(\varepsilon)(f(\varepsilon) \circ \boldsymbol{\Phi}^\varepsilon)(u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon).$$

Using expansions (18) and (21) and taking definition (4) into account, we obtain the asymptotic formula

$$\mathcal{P}(\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\, \mathcal{P}_0^{(n)}(\boldsymbol{\Phi}) \qquad [\mathcal{P}_0^{(0)}(\boldsymbol{\Phi}) = \mathcal{P}(0)] \tag{34}$$

with the corresponding derivatives of the energy function in the form

$$\mathcal{P}_0^{(1)}(\boldsymbol{\Phi}) \equiv \mathcal{P}_0'(\boldsymbol{\Phi}) = -\frac{1}{2}\int\limits_{\Omega_0} [F_1(\boldsymbol{\Phi})u^0 + f(0)\dot{u}(\boldsymbol{\Phi})], \tag{35}$$

$$\mathcal{P}_0^{(2)}(\boldsymbol{\Phi}) \equiv \mathcal{P}_0''(\boldsymbol{\Phi}) = -\frac{1}{2}\int\limits_{\Omega_0} [F_2(\boldsymbol{\Phi})u^0 + 2F_1(\boldsymbol{\Phi})\dot{u}(\boldsymbol{\Phi}) + f(0)\ddot{u}(\boldsymbol{\Phi})] \tag{36}$$

or, generally,

$$\mathcal{P}_0^{(n)}(\boldsymbol{\Phi}) = -\frac{1}{2}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \int\limits_{\Omega_0} F_{n-k}(\boldsymbol{\Phi}) \overset{(k)}{u}(\boldsymbol{\Phi}). \tag{37}$$

It follows from (18) and (37) that

$$\mathcal{P}(\varepsilon) - \sum_{k=0}^{n} \frac{\varepsilon^k}{k!}\mathcal{P}_0^{(k)}(\boldsymbol{\Phi}) = -\frac{1}{2}\sum_{k=0}^{n} \frac{\varepsilon^k}{k!} \int\limits_{\Omega_0} F_k(\boldsymbol{\Phi})\Big(u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon - \sum_{m=0}^{n-k} \frac{\varepsilon^m}{m!} \overset{(m)}{u}(\boldsymbol{\Phi})\Big)$$

$$- \frac{1}{2}\,\frac{\varepsilon^{n+1}}{(n+1)!} \int\limits_{\Omega_0} \bar{F}_{n+1}(\boldsymbol{\Phi})(u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon)$$

with the residual term $\bar{F}_{n+1}(\boldsymbol{\Phi})$ in expansion (18). Using estimates (12) and (26), we have

$$\Big|\mathcal{P}(\varepsilon) - \sum_{k=0}^{n} \frac{\varepsilon^k}{k!}\, \mathcal{P}_0^{(k)}(\boldsymbol{\Phi})\Big| \leqslant c\,\varepsilon^{n+1} \qquad (n = 0, 1, \ldots). \tag{38}$$

The order of the global derivatives of the solution included in formulas (35)–(37) can be reduced by unity. We set $v = \dot{u}(\boldsymbol{\Phi})$ in (1) and $v = u^0$ in (22). Then, we have

$$\int\limits_{\Omega_0} f(0)\dot{u}(\boldsymbol{\Phi}) = \int\limits_{\Omega_0} \nabla u^0 \cdot \nabla\dot{u}(\boldsymbol{\Phi}) = \int\limits_{\Omega_0} \nabla\dot{u}(\boldsymbol{\Phi}) \cdot \nabla u^0 = \int\limits_{\Omega_0} [F_1(\boldsymbol{\Phi})u^0 - A_1(\boldsymbol{\Phi}; u^0, u^0)];$$

therefore, Eq. (35) takes the equivalent form

$$\mathcal{P}_0'(\boldsymbol{\Phi}) = \int\limits_{\Omega_0} \Big[- F_1(\boldsymbol{\Phi})u^0 + \frac{1}{2}\, A_1(\boldsymbol{\Phi}; u^0, u^0)\Big]. \tag{39}$$

Assuming that $v = \ddot{u}(\boldsymbol{\Phi})$ in (1), $v = \dot{u}(\boldsymbol{\Phi})$ in (22), and $v = u^0$ in (23), we obtain the following formula from (36):

$$\mathcal{P}_0''(\boldsymbol{\Phi}) = \int\limits_{\Omega_0} \Big[-F_2(\boldsymbol{\Phi})u^0 + A_2(\boldsymbol{\Phi}; u^0, u^0) - |\nabla\dot{u}(\boldsymbol{\Phi})|^2\Big]. \tag{40}$$

Similarly, substitution of $v = \overset{(n)}{u}(\boldsymbol{\Phi})$ in (1), $v = \overset{(n-1)}{u}(\boldsymbol{\Phi})$ in (22), and $v = u^0$ in (24) yields the following relation for the nth order derivative from (37):

$$\mathcal{P}_0^{(n)}(\boldsymbol{\Phi}) = \int\limits_{\Omega_0} \Big(- F_n(\boldsymbol{\Phi})u^0 - \frac{1}{2}\sum_{k=1}^{n-1} F_{n-k}(\boldsymbol{\Phi}) \overset{(k)}{u}(\boldsymbol{\Phi}) - \frac{n}{2}\nabla \overset{(n-2)}{u}(\boldsymbol{\Phi}) \cdot \nabla\dot{u}(\boldsymbol{\Phi})$$

$$+ \frac{1}{2}\sum_{k=2}^{n} \frac{n!}{(n-k)!}\, A_k(\boldsymbol{\Phi}; \overset{(n-k)}{u}(\boldsymbol{\Phi}), u^0)\Big), \qquad n = 2, 3, \ldots\ . \tag{41}$$

Thus, we proved the following theorem:

Theorem 4. *For a small ε, derivatives of the potential energy $\mathcal{P}_0^{(n)}(\boldsymbol{\Phi})$ $(n = 1, 2, \ldots)$ with respect to a linear perturbation of the region exist that are defined by formulas* (35)–(37) *or* (39)–(41) *and give the asymptotic expansion* (34) *in ε of the potential energy function with estimate* (38).

In addition to Theorem 4, the definition of the potential energy leads to the following lemma:

Lemma 1. *If two different perturbations $\boldsymbol{\Phi}^\varepsilon = \boldsymbol{I} + \varepsilon\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}^\varepsilon = \boldsymbol{I} + \varepsilon\boldsymbol{\Psi}$ map a region with a crack Ω_0 onto the same perturbed region Ω_ε for any ε, then $\mathcal{P}_0^{(n)}(\boldsymbol{\Phi}) = \mathcal{P}_0^{(n)}(\boldsymbol{\Psi})$ for all n.*

Indeed, under the conditions of Lemma 1 for $\boldsymbol{\Phi}^\varepsilon$ and $\boldsymbol{\Psi}^\varepsilon$, we formulate the same perturbation problem (6); therefore, because of the uniqueness of its solution, we obtain the same function of the potential energy $\mathcal{P}(\varepsilon)$ from (7). Hence, from the corresponding expansions (34) derived for $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, we obtain the statement formulated as Lemma 1.

Furthermore, if the local expansion of solution (32) holds under conditions of Theorem 3 in a set $\overline{K} \subseteq \overline{\Omega}_0$ and $f(\varepsilon) \equiv 0$ in $\Omega_\varepsilon \setminus \overline{K}$ for all ε, then from definition (7) we obtain the following energy derivatives:

$$\mathcal{P}_0^{(n)}(\boldsymbol{\Phi}) = -\frac{1}{2}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\int\limits_K \frac{\partial^k f}{\partial\varepsilon^k}(0)\, u^{(n-k)}(\boldsymbol{\Phi}) \qquad (n = 1, 2, \ldots).$$

Let $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = 0$. Then, the first-order derivative of the potential energy is a linear continuous functional $\mathcal{L}_1$ with respect to $\boldsymbol{\Phi}$. By virtue of formulas (39), (19), and (15), it is written as

$$\mathcal{L}_1(\boldsymbol{\Phi}) = \int\limits_{\Omega_0}\Big[-\operatorname{div}(\boldsymbol{\Phi} f(0))u^0 + \frac{1}{2}\operatorname{div}\boldsymbol{\Phi}|\nabla u^0|^2 - \Big(\nabla u^0 : \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big)\cdot\nabla u^0\Big], \tag{42}$$

and the global first-order derivative of the solution $\dot{u}(\boldsymbol{\Phi})$ as a solution of the linear (in this case, with respect to $\boldsymbol{\Phi}$) problem (22) is also linear with respect to $\boldsymbol{\Phi}$. In addition, let $\partial^2 f/\partial\varepsilon^2\Big|_{\varepsilon=0} = 0$. Then, the second-order derivative of the energy function from representation (40) is associated with the quadratic potential $\mathcal{L}_2$, which, according to (16) and (20), can be represented in symmetric form:

$$\mathcal{L}_2(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = \int\limits_{\Omega_0}\Big\{-\Big[\Big(\Big|\frac{\partial(\Phi_1, \Psi_2)}{\partial\boldsymbol{x}}\Big| + \Big|\frac{\partial(\Psi_1, \Phi_2)}{\partial\boldsymbol{x}}\Big|\Big)f(0)$$

$$+\operatorname{div}\boldsymbol{\Phi}(\boldsymbol{\Psi}\cdot\nabla f(0)) + \operatorname{div}\boldsymbol{\Psi}(\boldsymbol{\Phi}\cdot\nabla f(0)) + \sum_{i,j=1}^{2}\Phi_i\Psi_j\frac{\partial^2 f(0)}{\partial x_i\partial x_j}\Big]u^0$$

$$-\frac{1}{2}\Big(\Big|\frac{\partial(\Phi_1, \Psi_2)}{\partial\boldsymbol{x}}\Big| + \Big|\frac{\partial(\Psi_1, \Phi_2)}{\partial\boldsymbol{x}}\Big|\Big)|\nabla u^0|^2 + \sum_{i,j,k=1}^{2}\Phi_{i,k}\Psi_{j,k}u^0_{,i}u^0_{,j} - \nabla\dot{u}(\boldsymbol{\Phi})\cdot\nabla\dot{u}(\boldsymbol{\Psi})\Big\}. \tag{43}$$

Thus, the following lemma is valid.

Lemma 2. *If $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = 0$, the first-order derivative of the potential energy $\mathcal{P}_0'(\boldsymbol{\Phi})$ is associated with the linear continuous functional $\mathcal{L}_1(\boldsymbol{\Phi})$ from formula (42). If $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = \partial^2 f/\partial\varepsilon^2\Big|_{\varepsilon=0} = 0$, the second-order derivative of the potential energy $\mathcal{P}_0''(\boldsymbol{\Phi})$ is associated with the bilinear symmetric continuous functional $\mathcal{L}_2(\boldsymbol{\Phi}, \boldsymbol{\Phi})$ from* (43).

1.5. Invariant Energy Integral. Let K be the region with a piecewise-smooth boundary ∂K and $\overline{K} \subseteq \overline{\Omega}_0$, in which the following conditions hold:

(a) the solution $u^0 \in H^2$ in $\overline{K}$;

(b) $f = 0$ or $\boldsymbol{\Phi} = 0$ in $\overline{\Omega} \setminus K$;

(c) $\boldsymbol{\Phi}(\boldsymbol{x})$ is such that the relation $(1/2)\operatorname{div}\boldsymbol{\Phi}|\boldsymbol{q}|^2 - (\boldsymbol{q} : \partial\boldsymbol{\Phi}/\partial\boldsymbol{x})\cdot\boldsymbol{q} = 0$ is valid for any vector $\boldsymbol{q} = (q_1, q_2)$ almost everywhere in $\Omega \setminus \overline{K}$.

The condition (c) is satisfied, for example, for the following vectors $\boldsymbol{\Phi}$: (c_1, c_2), $c(x_1, x_2)$, and $c(-x_2, x_1)$ and their linear combination (c, c_1, and c_2 are arbitrary constants). Let the conditions of Lemma 2 be satisfied. We consider the first-order derivative of the potential energy, which in this case is equal to $\mathcal{L}_1(\boldsymbol{\Phi})$ [see (42)]. By virtue of the assumptions (b) and (c), the integral from (42) over the region $\Omega_0 \setminus \overline{K}$ is zero, and, hence, the only integral over K is retained, which can be integrated by parts by virtue of the assumption (a).

As a result, we have the relation

$$\mathcal{L}_1(\boldsymbol{\Phi}) = \int\limits_K \frac{\partial u^0}{\partial \boldsymbol{\Phi}}(f(0) + \Delta u^0) + \int\limits_{\partial K} \Big[\frac{1}{2}(\boldsymbol{\Phi}\cdot\boldsymbol{\theta})|\nabla u^0|^2 - \frac{\partial u^0}{\partial \boldsymbol{\Phi}}\,\frac{\partial u^0}{\partial \boldsymbol{\theta}}\Big],$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)$ is an outward normal vector to the boundary ∂K. The integral over the region K vanishes by virtue of the equilibrium equation (2). Thus, we have the integral only over the contour ∂K:

$$I(\boldsymbol{\Phi}) = \int\limits_{\partial K} \Big[\frac{1}{2}\,(\boldsymbol{\Phi}\cdot\boldsymbol{\theta})|\nabla u^0|^2 - \frac{\partial u^0}{\partial \boldsymbol{\Phi}}\,\frac{\partial u^0}{\partial \boldsymbol{\theta}}\Big]. \tag{44}$$

Lemma 3. *Let* $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = 0$. *If there exists a set of regions* $\{K\}$ *for which the conditions* (a)–(c) *are satisfied, then the first-order derivative of the potential energy* $\mathcal{P}_0'(\boldsymbol{\Phi})$ *with respect to perturbation of the region has the form of the invariant integral* $I(\boldsymbol{\Phi})$ *from* (44) *over an arbitrary contour* ∂K*, and this integral is determined by selection of the perturbing function* $\boldsymbol{\Phi}$.

2. CRACK GROWTH

2.1. Perturbation of Local Shear along a Crack. Let a crack Γ_0 occupy a straight interval of length L with tips at the points $\boldsymbol{O} = (0, 0)$ and $\boldsymbol{C} = (L\tau_1, L\tau_2)$ [$\boldsymbol{\tau} = (\tau_1, \tau_2)$ is a directing tangent vector], and $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is a normal vector to Γ_0. We choose a patch function $\chi \in W^{1,\infty}(\mathbb{R}^2)$ that is finite with support $\operatorname{supp}\chi \subset \Omega$ and is equal to unity in some neighborhood $\overline{\mathcal{O}}$ of the crack tip $\boldsymbol{C}$ and the second tip lies outside $\operatorname{supp}\chi$. For example, if B_δ is a circle of radius δ with center at the point $\boldsymbol{C}$, we can choose $\chi(r) \equiv 1$ inside $B_{\delta/2}$ and $\chi(r) \equiv 0$ outside B_δ for $r = |\boldsymbol{x} - L\boldsymbol{\tau}|$.

Relations (5) with $\boldsymbol{\Phi} = \boldsymbol{\tau}\chi$ define a local shear transformation along the crack that maps Γ_0 onto a rectilinear crack Γ_ε of length $L+\varepsilon$ with tips at the points $\boldsymbol{O}$ and $((L+\varepsilon)\tau_1, (L+\varepsilon)\tau_2)$. The crack length is varied by varying the parameter ε. In this case, $\operatorname{div}\boldsymbol{\Phi} = \partial\chi/\partial\boldsymbol{\tau}$ and $|\partial\boldsymbol{\Phi}/\partial\boldsymbol{x}| = 0$. Then, according to Theorem 2, the global derivatives of the solution $\overset{(n)}{u}(\boldsymbol{\tau}\chi) \in \tilde{H}^1(\Omega_0)$ $(n = 1, 2, \ldots)$ are determined from problems (22)–(24), which in this case take the simpler form

$$\int\limits_{\Omega_0} \nabla\dot{u}(\boldsymbol{\tau}\chi)\cdot\nabla v = \int\limits_{\Omega_0} \Big[\Big(\frac{\partial f}{\partial\varepsilon}(0) + \frac{\partial}{\partial\boldsymbol{\tau}}\Big(\chi f(0)\Big)\Big)v - \frac{\partial\chi}{\partial\boldsymbol{\tau}}\,\nabla u^0\cdot\nabla v + \nabla\chi\cdot\Big(\nabla u^0\,\frac{\partial v}{\partial\boldsymbol{\tau}} + \nabla v\,\frac{\partial u^0}{\partial\boldsymbol{\tau}}\Big)\Big] \ \forall\, v \in \tilde{H}^1(\Omega_0); \tag{45}$$

$$\int\limits_{\Omega_0} \nabla \overset{(n)}{u}(\boldsymbol{\tau}\chi)\cdot\nabla v = \int\limits_{\Omega_0} \Big\{F_n(\boldsymbol{\tau}\chi)v + n\Big[\nabla\chi\cdot\Big(\nabla\overset{(n-1)}{u}(\boldsymbol{\tau}\chi)\,\frac{\partial v}{\partial\boldsymbol{\tau}} + \nabla v\,\frac{\partial\overset{(n-1)}{u}}{\partial\boldsymbol{\tau}}(\boldsymbol{\tau}\chi)\Big)$$

$$- \frac{\partial\chi}{\partial\boldsymbol{\tau}}\,\nabla\overset{(n-1)}{u}(\boldsymbol{\tau}\chi)\cdot\nabla v\Big] - |\nabla\chi|^2\sum_{k=2}^{n}\frac{n!}{(n-k)!}\Big(-\frac{\partial\chi}{\partial\boldsymbol{\tau}}\Big)^{k-2}\frac{\partial\overset{(n-k)}{u}}{\partial\boldsymbol{\tau}}(\boldsymbol{\tau}\chi)\,\frac{\partial v}{\partial\boldsymbol{\tau}}\Big\}. \tag{46}$$

By formal integration of Eq. (45) by parts, we obtain the following relations for the region Ω_0 and on the crack by virtue of (2) and (3):

$$-\Delta\dot{u}(\boldsymbol{\tau}\chi) = \frac{\partial f}{\partial\varepsilon}(0) - \Delta\Big(\chi\,\frac{\partial u^0}{\partial\boldsymbol{\tau}}\Big) \quad \text{in} \quad \Omega_0, \qquad \frac{\partial\dot{u}}{\partial\boldsymbol{\nu}}(\boldsymbol{\tau}\chi) = \frac{\partial\chi}{\partial\boldsymbol{\nu}}\,\frac{\partial u^0}{\partial\boldsymbol{\tau}} \quad \text{on} \quad \Gamma_0. \tag{47}$$

Similarly, from (46) for $n = 2, 3, \ldots$, we obtain the equalities

$$\begin{gathered}-\Delta\overset{(n)}{u}(\boldsymbol{\tau}\chi) = F_n(\boldsymbol{\tau}\chi) + n\Big[\frac{\partial\chi}{\partial\boldsymbol{\tau}}\,\Delta\overset{(n-1)}{u}(\boldsymbol{\tau}\chi) - 2\nabla\chi\cdot\nabla\Big(\frac{\partial\overset{(n-1)}{u}}{\partial\boldsymbol{\tau}}(\boldsymbol{\tau}\chi)\Big) \\ - \Delta\chi\frac{\partial\overset{(n-1)}{u}}{\partial\boldsymbol{\tau}}(\boldsymbol{\tau}\chi)\Big] + \sum_{k=2}^{n}\frac{n!}{(n-k)!}\,\frac{\partial}{\partial\boldsymbol{\tau}}\Big[|\nabla\chi|^2\Big(-\frac{\partial\chi}{\partial\boldsymbol{\tau}}\Big)^{k-2}\frac{\partial\overset{(n-k)}{u}}{\partial\boldsymbol{\tau}}(\boldsymbol{\tau}\chi)\Big], \\ \frac{\partial\overset{(n)}{u}}{\partial\boldsymbol{\nu}}(\boldsymbol{\tau}\chi) = n\Big(\frac{\partial\chi}{\partial\boldsymbol{\nu}}\,\frac{\partial\overset{(n-1)}{u}}{\partial\boldsymbol{\tau}}(\boldsymbol{\tau}\chi) - \frac{\partial\chi}{\partial\boldsymbol{\tau}}\,\frac{\partial\overset{(n-1)}{u}}{\partial\boldsymbol{\nu}}(\boldsymbol{\tau}\chi)\Big).\end{gathered} \tag{48}$$

It follows from representation (47) that if $u^0 \in H^2(\Omega_0)$, the solution of problem (45) is a function $\dot{u}(\boldsymbol{\tau}\chi) = \chi\partial u^0/\partial\boldsymbol{\tau} + u'$, where $u' \in \tilde{H}^1(\Omega_0)$ is a solution of the variational problem

$$\int\limits_{\Omega_0} \nabla u' \cdot \nabla v = \int\limits_{\Omega_0} \frac{\partial f}{\partial \varepsilon}(0) v \qquad \forall\, v \in \tilde{H}^1(\Omega_0).$$

In this case, according to definition (33), the first-order local derivative $u'(\boldsymbol{\tau}\chi)$ coincides with u', and for f that does not depend on ε, there exists a strong shape derivative that is identically equal to zero.

Following Lemma 1, the derivatives of the potential energy function for the perturbing function $\boldsymbol{\tau}\chi$ with respect to the crack length do not depend on the choice of the patch function χ and are determined in accordance with (39)–(41):

$$\mathcal{P}_0'(\boldsymbol{\tau}\chi) = \int\limits_{\Omega_0} \Big[-\Big(\frac{\partial f}{\partial \varepsilon}(0) + \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\chi f(0)\Big)\Big) u^0 + \frac{1}{2}\frac{\partial \chi}{\partial \boldsymbol{\tau}} |\nabla u^0|^2 - \frac{\partial u^0}{\partial \boldsymbol{\tau}} \nabla\chi \cdot \nabla u^0 \Big]. \tag{49}$$

The following derivatives are similar in form. Let $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = 0$. We set $K = (\Omega_0 \setminus \overline{\mathcal{O}}) \cap \operatorname{supp}\chi$. It can easily be shown that inside $\overline{K}$, the solution u^0 of problem (1) has additional H^2-smoothness, i.e., the condition (a) of Lemma 3 is satisfied. Outside $\operatorname{supp}\chi$, the transform $\boldsymbol{\tau}\chi \equiv 0$, and in $\overline{\mathcal{O}}$, $\boldsymbol{\tau}\chi \equiv \boldsymbol{\tau}$, and let $f(0) \equiv 0$. In this case, the conditions (b) in $\overline{\Omega} \setminus K$ and (c) in $\Omega \setminus \overline{K}$ are satisfied; therefore, equality (44) holds on the boundary consisting of the contour $\partial\mathcal{O}$ and the internal part of the crack $\Gamma_0 \cap \overline{K}$. Since, on the crack, $\boldsymbol{\theta} = \mp\boldsymbol{\nu}$, then, $\chi\boldsymbol{\tau} \cdot \boldsymbol{\theta} = 0$ and $\partial u^0/\partial\boldsymbol{\theta} = 0$ by virtue of the Neumann boundary condition (3). Thus, for $f(0) \equiv 0$ in the neighborhood of the crack tip, the first-order derivative of the potential energy with respect to the local shear perturbation along the crack $\mathcal{P}_0'(\boldsymbol{\tau}\chi)$ [see (49)] is the invariant energy integral (44) written as

$$I(\boldsymbol{\tau}\chi) = \int\limits_{\partial\mathcal{O}} \Big[\frac{1}{2}(\boldsymbol{\tau}\cdot\boldsymbol{\theta})|\nabla u^0|^2 - \frac{\partial u^0}{\partial \boldsymbol{\tau}}\frac{\partial u^0}{\partial \boldsymbol{\theta}}\Big], \tag{50}$$

where $\partial\mathcal{O}$ is any closed contour from this neighborhood, which might contain part of the crack. Formula (50) is known in fracture mechanics as the Cherepanov–Rice integral independent of the path of integrating.

2.2. Asymptotic Behavior of the Stress Intensity Factors. We use the global asymptotic method to determine the stress intensity factors (SIF). Let the crack tip $\boldsymbol{O}$ reach the external boundary $\partial\Omega$, so that we can consider only one crack tip $\boldsymbol{C}$. There is a well-known theorem (see [15]) representing the solution of problem (1) for a rectilinear crack as the sum of the singular and regular functions

$$u^0 = K(0)\chi(r)\sqrt{r}\sin(\varphi/2) + w^0, \tag{51}$$

where $w^0 \in H^2(\Omega_0)$ is a regular function and (r,φ) are polar coordinates in the neighborhood of $\boldsymbol{C}$, i.e., $\boldsymbol{x}\cdot\boldsymbol{\tau} - L = r\cos\varphi$ and $\boldsymbol{x}\cdot\boldsymbol{\nu} = r\sin\varphi$ ($|\varphi| \leqslant \pi$). In this case, the constant $K(0)$ in (51) is determined uniquely and is known in fracture mechanics as the stress intensity factor. To determine $K(0)$, we construct the auxiliary weight function

$$\zeta = (\chi(r)/(2\sqrt{r}))\sin(\varphi/2) + V, \tag{52}$$

where $V \in \tilde{H}^1(\Omega_0)$ is a unique solution of the variational problem [compare with (22) and (45)]

$$\int\limits_{\Omega_0} \nabla V \cdot \nabla v = -\int\limits_{\Omega_0} A_1\Big(\boldsymbol{\tau}\chi; \sqrt{r}\sin\frac{\varphi}{2}, v\Big) \qquad \forall\, v \in \tilde{H}^1(\Omega_0). \tag{53}$$

By virtue of

$$\Delta\Big(\sqrt{r}\sin\frac{\varphi}{2}\Big) = 0, \qquad \frac{\partial}{\partial\boldsymbol{\tau}}\Big(\sqrt{r}\sin\frac{\varphi}{2}\Big) = -\frac{1}{2\sqrt{r}}\sin\frac{\varphi}{2}, \qquad \frac{\partial\chi(r)}{\partial\boldsymbol{\nu}} = 0,$$

integration of Eq. (53) by parts yields the following relations similar to (47):

$$-\Delta V = \Delta\Big(\chi(r)\frac{1}{2\sqrt{r}}\sin\frac{\varphi}{2}\Big) \quad \text{in} \quad \Omega_0, \qquad \frac{\partial V}{\partial\boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma_0.$$

Therefore, $\zeta \not\equiv 0$ is a solution $\zeta \in L^2(\Omega_0)$ of the problem

$$\Delta\zeta = 0 \quad \text{in} \quad \Omega_0, \qquad \frac{\partial\zeta}{\partial\boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma_0. \tag{54}$$

Let the patch function $\chi(r)$ be such that $\chi(r) \equiv 1$ for $0 \leqslant r \leqslant \delta/2$, i.e., in the circle $B_{\delta/2}$. From representation (52) it follows that ζ is a function of the class H^1 outside the neighborhood $B_{\delta/2}$ (crack tip), and, hence, the Green formula can be applied to the region $\Omega_0 \setminus \overline{B}_{\delta/2}$. Taking (2), (3), and (54) into account, we obtain

$$\int\limits_{\Omega_0 \setminus B_{\delta/2}} f(0)\zeta = -\int\limits_{\Omega_0 \setminus B_{\delta/2}} \Delta u^0 \zeta = \int\limits_{\partial B_{\delta/2}} \Big(\frac{\partial \zeta}{\partial \boldsymbol{\theta}}\, u^0 - \frac{\partial u^0}{\partial \boldsymbol{\theta}}\, \zeta \Big),$$

where $\boldsymbol{\theta} = -(\cos\varphi, \sin\varphi)$ is a normal to the region boundary $\mathbb{R}^2 \setminus \overline{B}_{\delta/2}$. Substituting representations (51) and (52) into the last equation, we calculate the integral

$$\int\limits_{\Omega_0 \setminus B_{\delta/2}} f(0)\zeta = \frac{K(0)}{2} \int\limits_{|\varphi|<\pi} \sin^2 \frac{\varphi}{2}\, d\varphi + I_\delta = \frac{\pi}{2}\, K(0) + I_\delta, \tag{55}$$

where

$$I_\delta = \frac{\delta}{2} \int\limits_{|\varphi|<\pi} \Big[\frac{\partial V}{\partial \boldsymbol{\theta}}\, w^0 - \frac{\partial w^0}{\partial \boldsymbol{\theta}}\, V + \frac{\partial V}{\partial \boldsymbol{\theta}}\, K(0)\sqrt{r} \sin \frac{\varphi}{2} - V \frac{\partial}{\partial \boldsymbol{\theta}} \Big(K(0)\sqrt{r} \sin \frac{\varphi}{2} \Big)$$

$$+ w^0 \frac{\partial}{\partial \boldsymbol{\theta}} \Big(\frac{1}{2\sqrt{r}} \sin \frac{\varphi}{2} \Big) - \frac{\partial w^0}{\partial \boldsymbol{\theta}}\, \frac{1}{2\sqrt{r}} \sin \frac{\varphi}{2} \Big]_{r=\delta/2} d\varphi.$$

Because the integrand in I_δ has an integrable singularity for $r = 0$, it follows that $I_\delta \to 0$ as $\delta \to 0$. Thus, due to the boundedness of $f(0)$, passing to the limit in (55) as $\delta \to 0$, we obtain the following formula for the SIF:

$$K(0) = \frac{2}{\pi} \int\limits_{\Omega_0} f(0)\zeta. \tag{56}$$

For the global derivatives $\dot{u}(\boldsymbol{\tau}\chi), \ldots, \overset{(n)}{u}(\boldsymbol{\tau}\chi)$ as solutions of problems (45) and (46) similar to (1), the following representation in the form of (51) is also valid:

$$\overset{(n)}{u}(\boldsymbol{\tau}\chi) = K^{(n)} \chi(r) \sqrt{r} \sin(\varphi/2) + w^n, \qquad w^n \in H^2(\Omega_0), \quad n = 1, 2, \ldots\,. \tag{57}$$

We require additional smoothness of the patch $\chi \in W^{2,\infty}(\mathbb{R}^2)$. Then, according to (56), from relations (47) and (48) taking into account that $\partial\chi(r)/\partial\boldsymbol{\nu} = 0$ on Γ_0, we obtain the corresponding formulas for the coefficients $K^{(n)}$:

$$K^{(1)} = \frac{2}{\pi} \int\limits_{\Omega_0} \Big[\frac{\partial f}{\partial \varepsilon}(0) + \chi \frac{\partial f(0)}{\partial \boldsymbol{\tau}} - 2\nabla\chi \cdot \nabla \Big(\frac{\partial u^0}{\partial \boldsymbol{\tau}} \Big) - \Delta\chi\, \frac{\partial u^0}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau}\chi) \Big] \zeta,$$

$$K^{(n)} = \frac{2}{\pi} \int\limits_{\Omega_0} \Big\{ F_n(\boldsymbol{\tau}\chi) + n \Big[\frac{\partial \chi}{\partial \boldsymbol{\tau}}\, \Delta \overset{(n-1)}{u}(\boldsymbol{\tau}\chi) - 2\nabla\chi \cdot \nabla \Big(\frac{\partial \overset{(n-1)}{u}}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau}\chi) \Big) \tag{58}$$

$$-\Delta\chi\, \frac{\partial \overset{(n-1)}{u}}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau}\chi) \Big] + \sum_{k=2}^{n} \frac{n!}{(n-k)!} \frac{\partial}{\partial \boldsymbol{\tau}} \Big[|\nabla\chi|^2 \Big(-\frac{\partial \chi}{\partial \boldsymbol{\tau}} \Big)^{k-2} \frac{\partial \overset{(n-k)}{u}}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau}\chi) \Big] \Big\} \zeta, \quad n \geqslant 2.$$

The integrals in (58) are defined correctly because χ is twice differentiable, f is smooth, and $\nabla\chi \equiv 0$ in the neighborhood of the crack tip $B_{\delta/2}$, where the second derivatives of the solutions $u^0, \dot{u}(\boldsymbol{\tau}\chi), \ldots$ are not defined. Summing up representations (51) and (57) multiplied by $\varepsilon^n/n!$ over n and applying Theorem 2, from (21) we obtain a representation of the mapped solution of the perturbed problem (6) in the form

$$u^\varepsilon \circ \boldsymbol{\Phi}^\varepsilon = K^\varepsilon \chi(r) \sqrt{r} \sin(\varphi/2) + w^\varepsilon \qquad [w^\varepsilon \in H^2(\Omega_0)] \tag{59}$$

with the coefficient

$$K^\varepsilon = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\, K^{(n)} \qquad [K^{(0)} = K(0)]. \tag{60}$$

At the same time, u^ε as a solution of problem (6) inside Ω_ε is also written as the sum

$$u^\varepsilon = K(\varepsilon)\chi(r_y)\sqrt{r_y}\sin(\varphi_y/2) + W^\varepsilon, \qquad W^\varepsilon \in H^2(\Omega_\varepsilon), \tag{61}$$

where the polar coordinates r_y and φ_y are defined in the neighborhood of the tip $\boldsymbol{C}^\varepsilon = ((L+\varepsilon)\tau_1,\ (L+\varepsilon)\tau_2)$ of the perturbed crack Γ_ε, i.e., $\boldsymbol{y}\cdot\boldsymbol{\tau} - (L+\varepsilon) = r_y\cos\varphi_y$ and $\boldsymbol{y}\cdot\boldsymbol{\nu} = r_y\sin\varphi_y$ ($|\varphi_y| \leqslant \pi$). Applying the transformation $\boldsymbol{y} = \boldsymbol{x} + \varepsilon\boldsymbol{\tau}\chi(r)$ in the neighborhood $\mathcal{O}$ where $\chi \equiv 1$, we obtain $\boldsymbol{x}\cdot\boldsymbol{\tau} - L = r_y(x)\cos\varphi_y(\boldsymbol{x})$ and $\boldsymbol{x}\cdot\boldsymbol{\nu} = r_y(x)\sin\varphi_y(\boldsymbol{x})$, from which $r_y(\boldsymbol{x}) = r$, $\varphi_y(\boldsymbol{x}) = \varphi$ in $\mathcal{O}$. Therefore, from (61), we obtain the local representation $u^\varepsilon\circ\boldsymbol{\Phi}^\varepsilon = K(\varepsilon)\sqrt{r}\sin(\varphi/2) + W^\varepsilon(\boldsymbol{x}+\varepsilon\boldsymbol{\tau})$ in $\mathcal{O}$. Comparing this representation with Eq. (59), we have $K(\varepsilon) = K^\varepsilon$ by virtue of the uniqueness of the solution u^ε. Thus, we proved the following theorem:

Theorem 5. *If the representation* (51) *of the solution of problem* (1) *and the similar representation* (61) *of the solution of the perturbed problem* (6) *are valid, the SIF admits asymptotic expansion* (60) *with derivatives that can be determined from relations* (56) and (58) *for* $\chi \in W^{2,\infty}(\mathbb{R}^2)$.

2.3. Local Tensile Perturbation. For a rectilinear crack Γ_0 of length L with tips $\boldsymbol{O} = (0,0)$ and $\boldsymbol{C} = (L\tau_1, L\tau_2)$, we consider a local tensile perturbation with $\boldsymbol{\Phi} = \boldsymbol{x}\chi/L$, which maps Γ_0 onto a crack Γ_ε of length $L+\varepsilon$ similarly to the shear $\boldsymbol{\tau}\chi$ (see Sec. 2.1). Then,

$$\operatorname{div}\boldsymbol{\Phi} = \operatorname{div}\Big(\frac{\boldsymbol{x}}{L}\,\chi\Big), \qquad \Big|\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}}\Big| = \frac{1}{2}\operatorname{div}\Big(\frac{\boldsymbol{x}}{L^2}\,\chi^2\Big).$$

The global derivatives corresponding to tension $\overset{(n)}{u}(\boldsymbol{x}\chi/L) \in \tilde{H}^1(\Omega_0)$ ($n = 1,2,\ldots$) are obtained as solutions of problems (22)–(24) and the derivatives of the energy function $\mathcal{P}_0^{(n)}(\boldsymbol{x}\chi/L)$ ($n = 1,2,\ldots$) are calculated by the formulas (39)–(41) for $\boldsymbol{\Phi} = \boldsymbol{x}\chi/L$. In this case, by virtue of Lemma 1,

$$\mathcal{P}_0^{(n)}(\boldsymbol{x}\chi/L) = \mathcal{P}_0^{(n)}(\boldsymbol{\tau}\chi) \qquad (n = 1,2,\ldots). \tag{62}$$

We construct an invariant integral that corresponds to the perturbation $\boldsymbol{x}\chi/L$ and assume that $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = 0$ and $f \equiv 0$ in $\mathcal{O}$, where $\chi \equiv 1$. In $\mathcal{O}$, we have

$$\frac{1}{2}\operatorname{div}\Big(\frac{\boldsymbol{x}}{L}\Big) - \Big(\boldsymbol{q} : \frac{\partial}{\partial\boldsymbol{x}}\Big(\frac{\boldsymbol{x}}{L}\Big)\Big)\cdot\boldsymbol{q} = 0$$

for any vector $\boldsymbol{q} = (q_1, q_2)$. In addition, χ is a finite function in the neighborhood of the crack tip. Thus, according to Lemma 3, the valid representation of the first-order derivative $\mathcal{P}_0'(\boldsymbol{x}\chi/L)$ in the form of integral (44) over the contour consisting of $\partial\mathcal{O}$ and the interior of the crack $\Theta = \Gamma_0 \cap (\operatorname{supp}\chi \setminus \mathcal{O})$. The integrand in this part of the crack is bounded due to additional local smoothness of the solution u^0 outside the crack tip. At the same time, by virtue of Lemma 1, the derivative does not depend on the patch function χ, and, hence, we can pass to the limit $\operatorname{meas}\Theta \to 0$ and obtain

$$I\Big(\frac{\boldsymbol{x}}{L}\,\chi\Big) = \int\limits_{\partial\mathcal{O}}\Big[\frac{1}{2}\Big(\frac{\boldsymbol{x}}{L}\cdot\boldsymbol{\theta}\Big)|\nabla u^0|^2 - \Big(\frac{\boldsymbol{x}}{L}\cdot\nabla u^0\Big)\frac{\partial u^0}{\partial\boldsymbol{\theta}}\Big]. \tag{63}$$

In addition, by virtue of (62), we have $I(\boldsymbol{x}\chi/L) = I(\boldsymbol{\tau}\chi)$.

2.4. Branching Crack. Let a branching crack Γ_0 consist of N rectilinear intervals of length $L_i > 0$ ($i = 1,\ldots,N$) that intersect at a nonzero angle at the tip $\boldsymbol{O} = (0,0)$. The directing tangent vectors are denoted by $\boldsymbol{\tau}^i = (\tau_1^i, \tau_2^i)$ and the tips of the intervals are denoted by $\boldsymbol{C}^i = (L_i\tau_1^i, L_i\tau_2^i)$ ($i = 1,\ldots,N$). We choose finite patch functions $\chi_i \in W^{1,\infty}(\mathbb{R}^2)$ with disjoint supports in the vicinity of the corresponding tips $\boldsymbol{C}^i$ ($i = 1,\ldots,N$). We consider a perturbation of the crack in the form of a linear combination $\boldsymbol{\Phi} = \sum\limits_{i=1}^{N} a_i\boldsymbol{\tau}^i\chi_i$ with several unknown constants a_i ($i = 1,\ldots,N$). Then, the coordinate transformation (5) with a small parameter ε maps Γ_0 onto a branching crack Γ_ε that consists of N rectilinear intervals of length $L_i + \varepsilon a_i$ ($i = 1,\ldots,N$). The corresponding potential energy function admits the asymptotic expansion (34) in ε:

$$\mathcal{P}(\varepsilon) = \mathcal{P}(0) + \varepsilon\mathcal{P}_0'(\boldsymbol{\Phi}) + \varepsilon^2\mathcal{P}_0''(\boldsymbol{\Phi})/2 + o(\varepsilon^2). \tag{64}$$

We assume that $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = \partial^2 f/\partial\varepsilon^2\Big|_{\varepsilon=0} = 0$. Then, according to Lemma 2, the first derivative of the potential energy $\mathcal{P}_0'(\mathbf{\Phi})$ is represented as a linear, continuous (with respect to $\mathbf{\Phi}$) functional $\mathcal{L}_1(\mathbf{\Phi})$ from (42) and, in this case, $\mathcal{L}_1(\mathbf{\Phi}) = \sum\limits_{i=1}^{N} a_i \mathcal{L}_1(\boldsymbol{\tau}^i\chi_i)$, where

$$\mathcal{L}_1(\boldsymbol{\tau}^i\chi_i) = \int\limits_{\Omega_0} \Big(-\frac{\partial}{\partial\boldsymbol{\tau}^i}\,(\chi_i f(0))u^0 + \frac{1}{2}\,\frac{\partial\chi_i}{\partial\boldsymbol{\tau}^i}\,|\nabla u^0|^2 - \frac{\partial u^0}{\partial\boldsymbol{\tau}^i}\,\nabla\chi_i\cdot\nabla u^0\Big), \quad i = 1,\dots,N.$$

The second-order derivative $\mathcal{P}_0''(\mathbf{\Phi})$ is obtained in a similar manner, as a quadratic functional $\mathcal{L}_2(\mathbf{\Phi},\mathbf{\Phi})$ [see (43)]. For the branching crack, it is written as

$$\mathcal{L}_2(\mathbf{\Phi},\mathbf{\Phi}) = \sum_{i,j=1}^{N} a_i a_j \mathcal{L}_2(\boldsymbol{\tau}^i\chi_i, \boldsymbol{\tau}^j\chi_j).$$

Because the supports $\operatorname{supp}\chi_i$ and $\operatorname{supp}\chi_j$ do not intersect if $i \neq j$, the functional $\mathcal{L}_2(\mathbf{\Phi},\mathbf{\Phi})$ is written as

$$\mathcal{L}_2(\boldsymbol{\tau}^i\chi_i, \boldsymbol{\tau}^j\chi_j) = \int\limits_{\Omega_0} \Big\{\Big[-\frac{\partial}{\partial\boldsymbol{\tau}^i}\Big(\chi_i^2\,\frac{\partial f(0)}{\partial\boldsymbol{\tau}^i}\Big)u^0 + |\nabla\chi_i|^2\Big|\frac{\partial u^0}{\partial\boldsymbol{\tau}^i}\Big|^2\Big]\delta_{ij} - \nabla\dot{u}(\boldsymbol{\tau}^i\chi_i)\cdot\nabla\dot{u}(\boldsymbol{\tau}^j\chi_j)\Big\},$$

where $i,j = 1,\dots,N$ and δ_{ij} is the Kronecker symbol. Thus, Eq. (64) becomes

$$\mathcal{P}(\varepsilon) = \mathcal{P}(0) + \varepsilon\sum_{i=1}^{N} a_i\mathcal{L}_1(\boldsymbol{\tau}^i\chi_i) + \frac{\varepsilon^2}{2}\sum_{i,j=1}^{N} a_i a_j\mathcal{L}_2(\boldsymbol{\tau}^i\chi_i, \boldsymbol{\tau}^j\chi_j) + o(\varepsilon^2). \tag{65}$$

We determine the quantities $\varepsilon_i = \varepsilon a_i$ $(i = 1,\dots,N)$, which are variations of the crack length.

To describe the crack growth, we use the Griffith energy fraction criterion. The total energy E can be written as the sum of the potential energy $\mathcal{P}$ and the surface energy S, which according to the Griffith hypothesis is distributed uniformly over the crack with some constant density $\gamma > 0$:

$$S(\varepsilon) \equiv \int\limits_{\Gamma_\varepsilon}\gamma = S(0) + \gamma\varepsilon\sum_{i=1}^{N} a_i, \qquad S(0) \equiv \int\limits_{\Gamma_0}\gamma = \gamma\sum_{i=1}^{N} L_i. \tag{66}$$

We define an approximate quadratic function of the total energy T: $E(\varepsilon) = T(\varepsilon) + o(\varepsilon^2)$, which, according to (65) and (66), depends on N parameters $\varepsilon_1,\dots,\varepsilon_N$:

$$T(\varepsilon_1,\dots,\varepsilon_N) = S(0) + \sum_{i=1}^{N}\varepsilon_i\Big(\gamma + \mathcal{L}_1(\boldsymbol{\tau}^i\chi_i)\Big) + \frac{1}{2}\sum_{i,j=1}^{N}\varepsilon_i\varepsilon_j\mathcal{L}_2(\boldsymbol{\tau}^i\chi_i, \boldsymbol{\tau}^j\chi_j). \tag{67}$$

According to the Griffith criterion, the crack can only grow, i.e., the conditions $\varepsilon_i \geqslant 0$ $(i = 1,\dots,N)$ must be satisfied. Using the Newton method and minimizing the quadratic function in (67) over the unknown positive parameters $\varepsilon_1,\dots,\varepsilon_N$, we obtain the following system of algebraic variational inequalities

$$\varepsilon_i \geqslant 0, \quad \Big(\gamma + \mathcal{L}_1(\boldsymbol{\tau}^i\chi_i) + \sum_{j=1}^{N}\varepsilon_j\mathcal{L}_2(\boldsymbol{\tau}^i\chi_i, \boldsymbol{\tau}^j\chi_j)\Big)(\bar{\varepsilon} - \varepsilon_i) \geqslant 0 \quad \forall\,\bar{\varepsilon} \geqslant 0, \quad i = 1,\dots,N \tag{68}$$

which describes local growth of the branching crack. If $\gamma + \mathcal{L}_1(\boldsymbol{\tau}^i\chi_i) \geqslant 0$ for all $i = 1,\dots,N$, then $\varepsilon_1 = 0,\dots,\varepsilon_N = 0$ is a solution of (68). Hence, the crack is stationary. Therefore, the growth condition of the crack Γ_0 follows from (68): there exists i such that $\gamma + \mathcal{L}_1(\boldsymbol{\tau}^i\chi_i) < 0$. In this case, to seek the local crack growth from (68), it is necessary that the condition of solvability of the system of variational inequalities $\det\{\mathcal{L}_2(\boldsymbol{\tau}^i\chi_i, \boldsymbol{\tau}^j\chi_j)\}_{i,j=1}^{n} > 0$ $(n = 1,\dots,N)$ be satisfied. Particular cases of (68) are a two-parametric system that describes the growth of a rectilinear crack (two tips) and a variational inequality with one parameter for a rectilinear crack whose one tip reaches the external boundary $\partial\Omega$.

3. LOCATION OF A CRACK IN A BODY

3.1. Invariant Energy Integrals. We consider the general case of geometry of a crack Γ_0 (see Sec. 1.1). We choose a patch function $\eta \in W^{1,\infty}(\mathbb{R}^2)$ that is finite in Ω and set $\eta \equiv 1$ in the vicinity of the crack D, i.e., $\overline{\Gamma}_0 \subset \overline{D} \subset \operatorname{supp} \eta \subset \Omega$. We assume that $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = 0$. If we assume that $f(0) \equiv 0$ in $\overline{D}$, the condition (b) of Lemma 3 will be satisfied. The condition (a) is satisfied because of the local smoothness of u^0 outside the neighborhood of singular points of the crack (tip or salient point), and, hence, in $\Omega \setminus D$. We consider a shear perturbation of the crack in the arbitrary direction $\boldsymbol{p} = (p_1, p_2)$, i.e., $\boldsymbol{\Phi} = \boldsymbol{p}\eta$. Then, $\partial\boldsymbol{\Phi}/\partial\boldsymbol{x} \equiv 0$ in D by virtue of $\eta \equiv 1$. Hence, the condition (c) is satisfied. According to Lemma 3 and taking the finite nature of η into account, from (44) we obtain the invariant integral

$$I(\boldsymbol{p}\eta) = \int\limits_{\partial D} \Big(\frac{1}{2}\,(\boldsymbol{p}\cdot\boldsymbol{\theta})|\nabla u^0|^2 - \frac{\partial u^0}{\partial \boldsymbol{p}}\,\frac{\partial u^0}{\partial \boldsymbol{\theta}}\Big) \tag{69}$$

over an arbitrary closed contour ∂D in the neighborhood of the entire crack where $f(0) \equiv 0$. For tensile perturbation of the crack $\boldsymbol{\Phi} = \boldsymbol{x}\eta$, the condition (c) is satisfied by virtue of $\operatorname{div}\boldsymbol{\Phi} = 2$, $\boldsymbol{q} : \partial\boldsymbol{\Phi}/\partial\boldsymbol{x} = \boldsymbol{q}\ \ \forall\, \boldsymbol{q}$ in D, and the following invariant integral holds:

$$I(\boldsymbol{x}\eta) = \int\limits_{\partial D} \Big(\frac{1}{2}(\boldsymbol{x}\cdot\boldsymbol{\theta})|\nabla u^0|^2 - (\boldsymbol{x}\cdot\nabla u^0)\frac{\partial u^0}{\partial \boldsymbol{\theta}}\Big). \tag{70}$$

Similarly, for perturbation of the linearized rotation of the crack $\boldsymbol{\Phi} = (-x_2, x_1)\eta$, the condition $\operatorname{div}\boldsymbol{\Phi} = 0$, $(\boldsymbol{q} : \partial\boldsymbol{\Phi}/\partial\boldsymbol{x})\cdot\boldsymbol{q} = (q_2, -q_1)\cdot(q_1, q_2) = 0\ \forall\, \boldsymbol{q}$ in D is satisfied, from which we obtain the integral

$$I((-x_2, x_1)\eta) = \int\limits_{\partial D} \Big(\frac{1}{2}(x_1\theta_2 - x_2\theta_1)|\nabla u^0|^2 - (x_1 u^0_{,2} - x_2 u^0_{,1})\frac{\partial u^0}{\partial \boldsymbol{\theta}}\Big). \tag{71}$$

Thus, we proved the following theorem:

Theorem 6. *Let* $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = 0$. *For a rectilinear crack, the first-order derivative of the potential energy with respect to the crack length is represented as equal invariant integrals for local shear along the crack* $I(\boldsymbol{\tau}\chi)$ (50) *and local tension* $I(\boldsymbol{x}\chi/L)$ (63) *along any closed contour in the neighborhood of the crack tip where* $f(0) \equiv 0$. *In the neighborhood of the entire crack where* $f(0) \equiv 0$, *for the crack perturbed by shear in an arbitrary direction, tension, and linearized rotation, the following invariant integrals are valid:* $I(\boldsymbol{p}\eta)$, $I(\boldsymbol{x}\eta)$, *and* $I((-x_2, x_1)\eta)$ [see (69), (70), and (71)], *respectively.*

3.2. Problem of Optimization of the Crack Shape. We consider a rectilinear crack Γ_0 of length L and formulate a linearized problem of optimization of the location of the crack Γ_0 in the region Ω with the potential energy $\mathcal{P}$ as a cost functional. Let η, as above, be a patch function finite in Ω, $\eta \equiv 1$ in $\overline{D}$, and $\overline{\Gamma}_0 \subset D$. The location of the crack in D can be described as a linear combination of shear and rotation. Instead of rotation, one can use linearized rotation and compensate for the variation in length due to tension. Thus, according to Theorem 6, we consider the linear combination of the four basic perturbations of the crack with the unknown constants $a_1, \ldots, a_4$:

$$\boldsymbol{\Phi} = \sum_{i=1}^{4} a_i \boldsymbol{\Phi}^i, \quad \boldsymbol{\Phi}^1 = (1,0)\eta, \quad \boldsymbol{\Phi}^2 = (0,1)\eta, \quad \boldsymbol{\Phi}^3 = (-x_2, x_1)\eta, \quad \boldsymbol{\Phi}^4 = \boldsymbol{x}\eta. \tag{72}$$

For a small parameter ε, we use (72) as a perturbation function for the linear coordinate transformation (5) of the region Ω_0 with a fixed rectilinear crack Γ_0 of length L in D. This transformation also maps Γ_0 onto a rectilinear crack Γ_ε of length

$$|\Gamma_\varepsilon| = L\sqrt{(\varepsilon a_3)^2 + (1 + \varepsilon a_4)^2}. \tag{73}$$

According to Theorem 4, we have an asymptotic expansion of the potential energy function (64) in ε. Let $\partial f/\partial\varepsilon\Big|_{\varepsilon=0} = \partial^2 f/\partial\varepsilon^2\Big|_{\varepsilon=0} = 0$. Then, in accordance with Lemma 2 and by virtue of linearity of (72), Eq. (64) can be represented as

$$\mathcal{P}(\varepsilon) = \mathcal{P}(0) + \varepsilon \sum_{i=1}^{4} a_i \mathcal{L}_1(\boldsymbol{\Phi}^i) + \frac{\varepsilon^2}{2} \sum_{i,j=1}^{4} a_i a_j \mathcal{L}_2(\boldsymbol{\Phi}^i, \boldsymbol{\Phi}^j) + o(\varepsilon^2). \tag{74}$$

In this case, we require that the initial crack length L be unchanged for any ε, which, in view of equality (73), leads to the condition

$$\varepsilon a_4 = \sqrt{1 - (\varepsilon a_3)^2} - 1 = -(\varepsilon a_3)^2/2 + o(\varepsilon^2). \tag{75}$$

We substitute (75) into (74) and omit terms of order $o(\varepsilon^2)$. As a result, we obtain a quadratic (with respect to ε) approximation P of the energy $\mathcal{P}$, which is a function of three parameters $\varepsilon_i = \varepsilon a_i$ $(i = 1, 2, 3)$:

$$P(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \mathcal{P}(0) + \sum_{i=1}^{3} \varepsilon_i \mathcal{L}_1(\mathbf{\Phi}^i) + \frac{1}{2} \sum_{i,j=1}^{3} \varepsilon_i \varepsilon_j \mathcal{L}_2(\mathbf{\Phi}^i, \mathbf{\Phi}^j) - \frac{\varepsilon_3^2}{2} \mathcal{L}_1(\mathbf{\Phi}^4). \tag{76}$$

According to the general variational principle, to find an optimal location of a crack in a body, the approximate function of the potential energy (76) can be minimized over $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^3$. After minimizing, we obtain a system of three linear equations for the unknown parameters $\varepsilon_1, \varepsilon_2$, and ε_3:

$$\mathcal{L}_1(\mathbf{\Phi}^i) + \sum_{j=1}^{3} \varepsilon_j \mathcal{L}_2(\mathbf{\Phi}^i, \mathbf{\Phi}^j) - \delta_{i3} \varepsilon_3 \, \mathcal{L}_1(\mathbf{\Phi}^4) = 0 \qquad (i = 1, 2, 3). \tag{77}$$

The inequality $\det \{\mathcal{L}_2(\mathbf{\Phi}^i, \mathbf{\Phi}^j) - \delta_{i3} \mathcal{L}_1(\mathbf{\Phi}^4)\}_{i,j=1}^{n} > 0$ $(n = 1, 2, 3)$ is condition of solvability of system (77). Relations (77) are a linearized model that is valid only for small ε_1, ε_2, and ε_3.

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